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Probability Theory and Applications (MA208)
Problem Sheet - 3

Conditional Probability and Independence

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1. Urn 1 contains x white and y red balls. Urn 2 contains z white and v red balls. A ball is chosen at random from urn 1 and put into urn 2. Then a ball is chosen at random from urn 2. What is the probability that this ball is white?

2. Two defective tubes get mixed up with two good ones. The tubes are tested, one by one, until both defectives are found.
   (a) What is the probability that the last defective tube is obtained on the second test?
   (b) What is the probability that the last defective tube is obtained on the third test?
   (c) What is the probability that the last defective tube is obtained on the fourth test?
   (d) Add the numbers obtained for (a), (b), and (c) above. Is the result surprising?

3. A box contains 4 bad and 6 good tubes. Two are drawn out together. One of them is tested and found to be good. What is the probability that the other one is also good?

4. A box contains 4 bad and 6 good tubes. The tubes are checked by drawing a tube at random, testing it and repeating the process until all 4 bad tubes are located. What is the probability that the fourth bad tube will be located
   (a) on the fifth test ?
   (b) on the tenth test ?

5. Suppose that A and B are independent events associated with an experiment. If the probability that A or B occurs equals 0.6, while the probability that A occurs equals 0.4, determine the probability that B occurs.

6. Twenty items, 12 of which are defective and 8 nondefective, are inspected one after the other. If these items are chosen at random, what is the probability that:
   (a) the first two items inspected are defective?
   (b) the first two items inspected are nondefective?
   (c) among the first two items inspected there is one defective and one nondefective?

7. Suppose that we have two urns, 1 and 2, each with two drawers. Urn 1 has a gold coin in one drawer and a silver coin in the other drawer, while urn 2 has a gold coin in each drawer. One urn is chosen at random; then a drawer is chosen at random from the chosen urn. The coin found in this drawer turns out to be gold. What is the probability that the coin came from urn 2?

8. A bag contains three coins, one of which is coined with two heads while the other two coins are normal and not biased. A coin is chosen at random from the bag and tossed four times in succession. If heads turn up each time, what is the probability that this is the two-headed coin?

9. In a bolt factory, machines A, B, and C manufacture $25, 35$, and 40 percent of the total output, respectively. Of their outputs, $5, 4$, and 2 percent, respectively, are defective bolts. A bolt is chosen at random and found to be defective. What is the probability that the bolt came from machine A? B? C?

10. Let A and B be two events associated with an experiment. Suppose that $P(A) = 0.4$ while $P(A \cup B) = 0.7$. Let $P(B) = p$.

    (a) For what choice of p are A and B mutually exclusive?

    (b) For what choice of p are A and B independent?

11. Three components of a mechanism, say C_1, C_2, and C_3 are placed (in series in a straight line). Suppose that these mechanisms are arranged in a random order. Let R be the event (C_2 is to the right of C_1), and let S be the event (C_3 is to the right of C_1). Are the events R and S independent? Why?

12. A die is tossed, and independently, a card is chosen at random from a regular deck. What is the probability that:

    (a) the die shows an even number and the card is from a red suit?

    (b) the die shows an even number or the card is from a red suit?

13. **Binary Number Problem :** A binary number is one composed only of the digits zero and one. (For example $1011, 1100$, etc.) These numbers play an important role in the use of electronic computers. Suppose that a binary number is made up of n digits. Suppose that the probability of an incorrect digit appearing is p and that errors in different digits are independent of one another. What is the probability of forming an incorrect number?

14. A die is thrown n times. What is the probability that "6" comes up at least once in the n throws ?

15. Each of two persons tosses three fair coins. What is the probability that they obtain the same number of heads ?

16. Two dice are rolled. Given that the faces show different numbers, what is the probability that one face is 4?

17. It is found that in manufacturing a certain article, defects of one type occur with probability 0.1 and defects of a second type with probability 0.05. (Assume independence between types of defects.) What is the probability that:

    (a) an article does not have both kinds of defects ?

    (b) an article is defective?

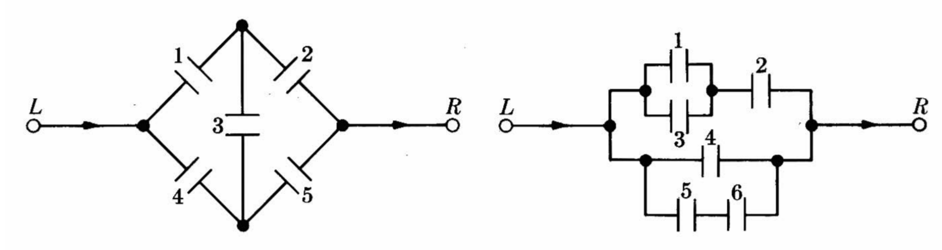
    (c) an article has only one type of defect, given that it is defective ?

18. The n events $A_1, A_2, \ldots, A_n$ are mutually independent if and only if we have for $k = 2, 3, \ldots, n$,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_n}).$$

    Verify that the number of conditions listed in the above is given by $2^n - n - 1$.

19. Prove that if A and B are independent events, so are A and $\bar{B}$, $\bar{A}$ and B, $\bar{A}$ and $\bar{B}$.



(a) (b)

20. **Electrical Circuit Problem :** In the above figure, assume that the probability of each relay being closed is p and that each relay is open or closed independently of any other relay. In each case find the probability that current flows from L to R.

Number of breakdowns	0	1	2	3	4	5	6
A	0.1	0.2	0.3	0.2	0.09	0.07	0.04
B	0.3	0.1	0.1	0.1	0.1	0.15	0.15

21. Two machines, A, B, being operated independently, may have a number of breakdowns each day. The above table gives the probability distribution of breakdowns for each machine. Compute the following probabilities.
    (a) A and B have the same number of breakdowns.
    (b) The total number of breakdowns is less than 4 ; less than 5.
    (c) A has more breakdowns than B.
    (d) B has twice as many breakdowns as A.
    (e) B has 4 breakdowns, when it is known that B has at least 2 breakdowns.
    (f) The minimum number of breakdowns of the two machines is 3; is less than 3.
    (g) The maximum number of breakdowns of the machines is 3; is more than 3.

22. Show that for fixed $A, P(B|A)$ satisfies the various postulates for probability.

23. If each element of a second order determinant is either zero or one, what is the probability that the value of the determinant is positive? (Assume that the individual entries of the determinant are chosen independently, each value being assumed with probability $\frac{1}{2}$.)

24. Show that the multiplication theorem $P(A \cap B) = P(A|B)P(B)$, established for two events, may be generalized to three events as follows:

$$P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C).$$

25. **Electrical Problem :** An electronic assembly consists of two subsystems, say A and B. From previous testing procedures, the following probabilities are assumed to be known:

$$P(A \text{ fails}) = 0.20,$$

$$P(B \text{ fails alone}) = 0.15,$$

$$P(A \text{ and fail}) = 0.15.$$

Evaluate the following probabilities :

(a) $P(A \text{ fails } | B \text{ has failed})$,

(b) $P(A \text{ fails alone})$.

26. Suppose that a large number of containers of candy are made up of two types, say A and B. Type A contains 70 percent sweet and 30 percent sour ones while B these percentages are reversed. Furthermore, suppose that 60 percent candy jars are of type A while the remainder are of type B.

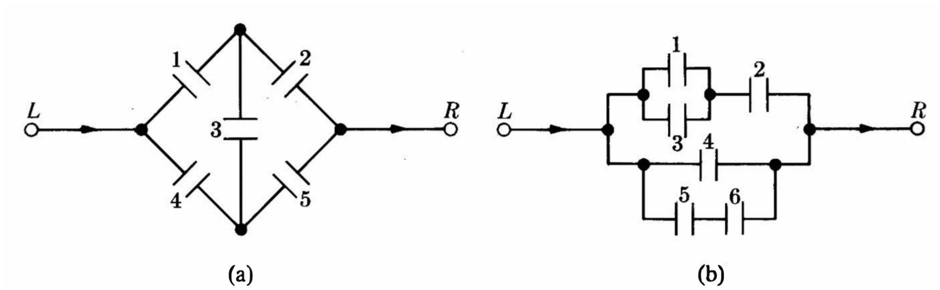
    Analyze the above example by deciding which of the types of candy jar, A or B, is involved, based on the evidence of two pieces of candy which were sampled.

27. Whenever an experiment is performed, the occurrence of a particular event A equals 0.2. The experiment is repeated, independently, until A occurs. Compute the probability that it will be necessary to carry out a fourth experiment.

28. Suppose that a mechanism has N tubes, all of which are needed for its functioning. To locate a malfunctioning tube one replaces each tube, successively, with a new one. Compute the probability that it will be necessary to check N tubes if the (constant) probability is p that a tube is out of order.

29. Prove: If $P(A|B) > P(A)$ then $P(B|A) > P(B)$.

30. A vacuum tube may come from any one of three manufacturers with probabilities $p_1 = 0.25, p_2 = 0.50$, and $P_3 = 0.25$. The probabilities that the tube will function properly during a specified period of time equal 0.1, 0.2, and 0.4, respectively, for the three manufacturers. Compute the probability that a randomly chosen lube will function for the specified period of time.



31. An electrical system consists of two switches of type A, one of type B, and four of type C, connected as in the above figure. Compute the probability that a break in the circuit cannot be eliminated with key K if the switches A, B, and C are open (i.e., out of order) with probabilities $0.3, 0.4$, and 0.2, respectively, and if they operate independently.

32. The probability that a system becomes overloaded is 0.4 during each run of an experiment. Compute the probability that the system will cease functioning in three independent trials of the experiment if the probabilities of failure in $1, 2$, or 3 trials equal $0.2, 0.5$, and 0.8, respectively.

33. Four radio signals are emitted successively. If the reception of any one signal is independent of the reception of another and if these probabilities are $0.1, 0.2, 0.3$, and 0.4, respectively, compute the probability that k signals will be received for $k = 0, 1, 2, 3, 4$.

34. The following (somewhat simple-minded) weather forecasting is used by an amateur forecaster. Each day is classified as "dry" or "wet" and the probability that any given day is the same as the preceding one is assumed to be a constant $p (0 < p < 1)$. Based on past records, it is supposed that January 1 has a probability of β of being "dry." Letting β_n = probability (the nth day of the year is "dry"),

obtain an expression for β, in terms of β and p. Also evaluate $\lim_{n\to\infty} \beta_n$ and interpret your result. (Hint: Express β_n in terms of β_{n-1}.)

35. Three newspapers, A, B, and C, are published in a city and a recent survey of readers indicates the following: 20 percent read A, 16 percent read B, 14 percent read C, 8 percent read A and B, 5 percent read A and C, 2 percent read A, B, and C, and 40 percent read B and C. For one adult chosen at random, compute the probability that

    (a) he reads none of the papers

    (b) he reads exactly one of the papers

    (c) he reads at least A and B if it is known that he reads at least one of the papers published.

36. A fair coin is tossed $2n$ times.

    (a) Obtain the probability that there will be an equal number of heads and tails.

    (b) Show that the probability computed in (a) is a decreasing function of n.

37. Urn 1, Urn 2, . . ., Urn n each contain α white and β black balls. One ball is taken from Urn 1 into Urn 2 and then one is taken from Urn 2 into Urn 3, etc. Finally, one ball is chosen from Urn n. If the first ball transferred was white, what is the probability that the last ball chosen is white? What happens as $n \to \infty$? [*Hint:* Let $p_n =$ Prob (nth ball transferred is white) and express p_n in terms of p_{n-1}.]

38. Urn 1 contains α white and β black balls while Urn 2 contains β white and α black balls. One ball is chosen (from one of the urns) and is then returned to that urn. If the chosen ball is white, choose the next ball from Urn 1; if the chosen ball is black, choose the next one from Urn 2. Continue in this manner. Given that the first ball chosen came from Urn 1, obtain Prob (nth ball chosen is white) and also the limit of this probability as $n \to \infty$.

39. A printing machine can print n "letters," say $\alpha_1, \ldots, \alpha_n$. It is operated by electrical impulses, each letter being produced by a *different* impulse. Assume that there exists a constant probability p of printing the correct letter and also assume independence. One of the n impulses, chosen at random, was fed into the machine twice and both times the letter α_1 was printed. Compute the probability that the impulse chosen was meant to print α_1.

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